Cheeger constant and Eigenvalues on Networks

E. Bendito, A. Carmona, A. M. Encinas y J.M. Gesto

angeles.carmona@upc.edu

Departament de Matemàtica Aplicada III. UPC

Notations and definitions

- (Γ, c, ν) a weighted network

Notations and definitions

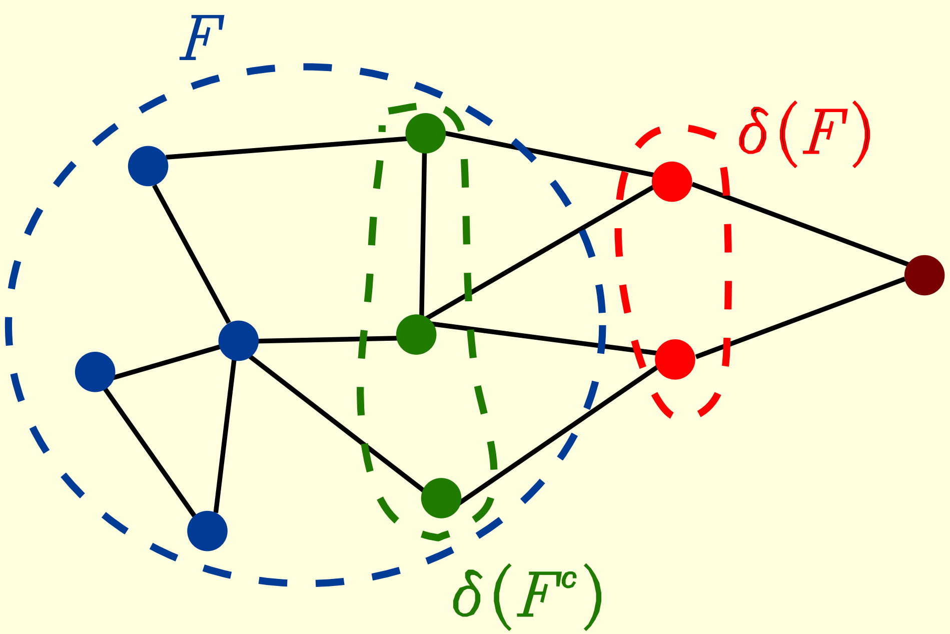
- (Γ, c, ν) a weighted network
- Γ a simple connected graph
- $c \in \mathcal{C}(V \times V)$ symmetric and $c(x,y) > 0$ iff $x \sim y$
- $\nu \in \mathcal{C}(V)$ $\nu(x) > 0, x \in V$

Notations and definitions

- (Γ, c, ν) a weighted network
- If $u \in \mathcal{C}(F)$: $||u||_{1,\nu} = \int_F |u|\nu \, dx \quad ||u||_{2,\nu} = \left(\int_F u^2 \nu \, dx\right)^{\frac{1}{2}}$

Notations and definitions

- (Γ, c, ν) a weighted network
- If $u \in \mathcal{C}(F)$: $||u||_{1,\nu} = \int_F |u|\nu\, dx \quad ||u||_{2,\nu} = \left(\int_F u^2 \nu\, dx\right)^{\frac{1}{2}}$
- If $F \subset V$: $\text{vol}_\nu(F) = ||\chi_F||_{1,\nu} \quad \text{vol}_c(\partial F) = \int_{\delta(F^c)\times\delta(F)} c(x,y)\, dxdy$



Cheeger constant

- Cheeger constant:

$$h_{\nu,c}(\Gamma) = \inf \left\{ \frac{\mathsf{vol}_c(\partial F)}{\mathsf{vol}_\nu(F)}; \, 0 < \mathsf{vol}_\nu(F) \leq \mathsf{vol}_c(V)/2 \right\}$$

Cheeger constant

- Cheeger constant:

$$h_{\nu,c}(\Gamma) = \inf \left\{ \frac{\mathsf{vol}_c(\partial F)}{\mathsf{vol}_\nu(F)}; \, 0 < \mathsf{vol}_\nu(F) \leq \mathsf{vol}_c(V)/2 \right\}$$

✓ Isoperimetric number: $c(x,y) = 1$ and $\nu = 1$

Cheeger constant

- Cheeger constant:

$$h_{\nu,c}(\Gamma) = \inf \left\{ \frac{\mathsf{vol}_c(\partial F)}{\mathsf{vol}_\nu(F)}; \, 0 < \mathsf{vol}_\nu(F) \leq \mathsf{vol}_c(V)/2 \right\}$$

✓ Isoperimetric number: $c(x, y) = 1$ and $\nu = 1$

✓ Cheeger constant: $c(x, y) = 1$ and $\nu = k$

Cheeger constant

- Cheeger constant:

$$h_{\nu,c}(\Gamma) = \inf \left\{ \frac{\mathsf{vol}_c(\partial F)}{\mathsf{vol}_\nu(F)}; 0 < \mathsf{vol}_\nu(F) \leq \mathsf{vol}_c(V)/2 \right\}$$

✓ Isoperimetric number: $c(x,y) = 1$ and $\nu = 1$

✓ Cheeger constant: $c(x,y) = 1$ and $\nu = k$

✓ Weighted Cheeger constant: $\nu(x) = \sum_{y \sim x} c(x,y)$

Laplacian and normal derivative

- The Laplace operator of (Γ, c, ν), $\quad \mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \int_V c(x,y)\,(u(x) - u(y))\,dy, \quad x \in V$$

Laplacian and normal derivative

- The Laplace operator of (Γ, c, ν), $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \int_V c(x,y)\,(u(x) - u(y))\,dy, \quad x \in V$$

- The conormal derivative of $u \in \mathcal{C}(\bar{F})$,

$$\frac{\partial u}{\partial \eta_F}(x) = \frac{1}{\nu(x)} \int_{\delta(F^c)} c(x,y)\,(u(x) - u(y))\,dy, \quad x \in \delta(F)$$

Laplacian and normal derivative

- The Laplace operator of (Γ, c, ν), $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \int_V c(x,y)\,(u(x) - u(y))\,dy, \quad x \in V$$

- The conormal derivative of $u \in \mathcal{C}(\bar{F})$,

$$\frac{\partial u}{\partial \eta_F}(x) = \frac{1}{\nu(x)} \int_{\delta(F^c)} c(x,y)\,(u(x) - u(y))\,dy, \quad x \in \delta(F)$$

- Green's identity, $u, v \in \mathcal{C}(\bar{F})$

$$\int_F v\mathcal{L}u\,\nu\,dx - \int_F u\mathcal{L}v\,\nu\,dx = \int_{\delta(F)} u\frac{\partial v}{\partial \mathsf{n}_F}\,\nu\,dx - \int_{\delta(F)} v\frac{\partial u}{\partial \mathsf{n}_F}\,\nu\,dx,$$

How to bound ?

- Take $v = \chi_F$ in Green's Identity

$$-\int_F \mathcal{L}u = \int_{F\times F^c} c(x,y)(u(y)-u(x))$$

How to bound ?

- Take $v = \chi_F$ in Green's Identity

$$-\int_F \mathcal{L}u = \int_{F\times F^c} c(x,y)(u(y)-u(x))$$

- Equilibrium measure of F

$$\mathcal{L}\gamma^F(x) = 1, \quad x \in F$$

$$\gamma^F(x) = 0, \qquad x \in F^c$$

Bounds on Cheeger constant

$$\begin{aligned}
\mathsf{vol}_\nu(F) &= \int_F \mathcal{L}\gamma^F \nu\, dx = -\int_{\delta(F)} \frac{\partial \gamma^F}{\partial \mathsf{n}_F} \nu\, dx \\
&= \int_{\delta(F)\times\delta(F^c)} c(x,y)\gamma^F(y) dx dy \\
&\leq \mathsf{vol}_c(\partial F)\, \gamma^F_M
\end{aligned}$$

- $\gamma^F_M = \max\left\{\gamma^F(x);\ x \in \delta(F^c)\right\}$
- $\gamma^F_m = \min\left\{\gamma^F(x);\ x \in \delta(F^c)\right\}$

Bounds on Cheeger constant

$\Longrightarrow$ $\mathrm{vol}_c(\partial F)\gamma_m^F \leq \mathrm{vol}_\nu(F) \leq \mathrm{vol}_c(\partial F)\gamma_M^F$

$\Longrightarrow$ With equality iff γ^F is constant on $\delta(F^c)$

Bounds on Cheeger constant

$\Longrightarrow$ $\text{vol}_c(\partial F)\gamma_m^F \leq \text{vol}_\nu(F) \leq \text{vol}_c(\partial F)\gamma_M^F$

$\Longrightarrow$ With equality iff γ^F is constant on $\delta(F^c)$

$$\min_{\substack{F \subset V \\ \text{vol}(F) \leq \frac{\text{vol}(V)}{2}}} \min_{x \in \delta(F^c)} \left\{ \frac{1}{\gamma^F(x)} \right\} \leq h(\Gamma) \leq \min_{\substack{F \subset V \\ \text{vol}(F) \leq \frac{\text{vol}(V)}{2}}} \max_{x \in \delta(F^c)} \left\{ \frac{1}{\gamma^F(x)} \right\}$$

Bounds on Cheeger constant

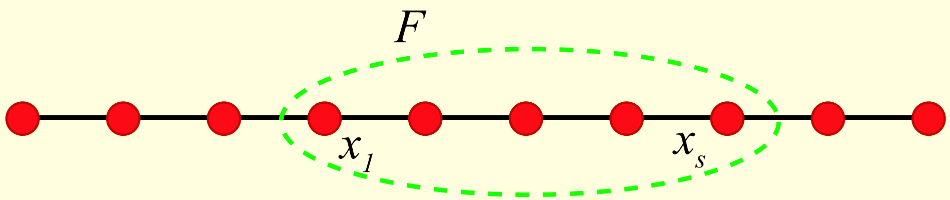
$$\Longrightarrow \quad \text{vol}_c(\partial F)\gamma_m^F \leq \text{vol}_\nu(F) \leq \text{vol}_c(\partial F)\gamma_M^F$$

$\Longrightarrow$ With equality iff γ^F is constant on $\delta(F^c)$

$$\Longrightarrow h(\Gamma) = \min\left\{ \frac{\displaystyle\int_{\delta(F)} \frac{\partial \chi_F}{\partial \boldsymbol{\eta}_F} d\nu}{\displaystyle\int_{\delta(F)} \frac{\partial \gamma^F}{\partial \boldsymbol{\eta}_F} d\nu};\ \ 0 < \text{vol}_\nu(F) \leq \text{vol}_\nu(V)/2 \right\}$$

Some examples

✓ The path on n vertices P_n : $F = \{x_1, \ldots, x_s\}$



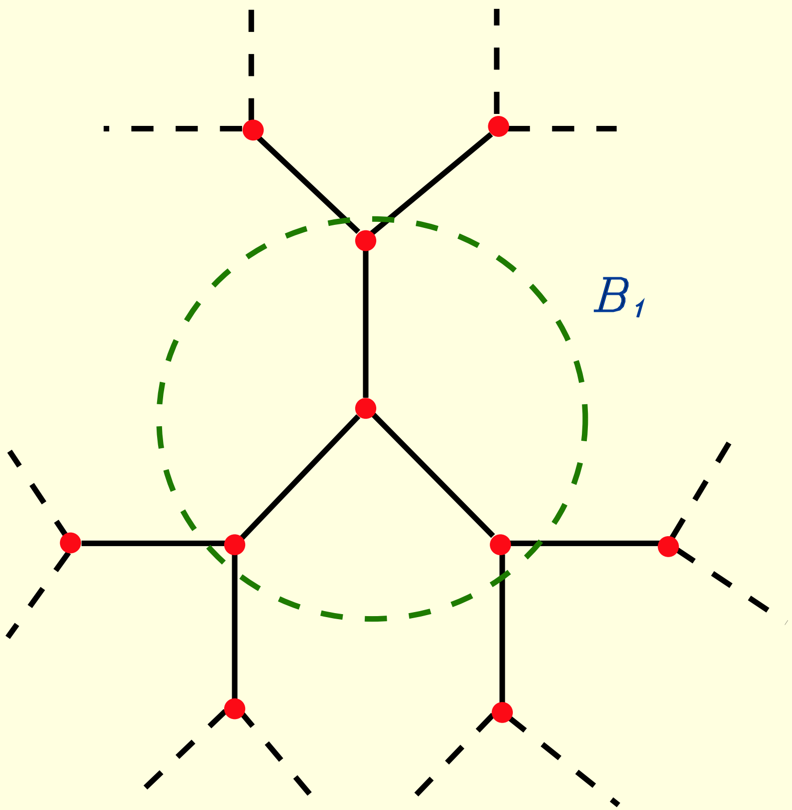
$$\circ\ \gamma^F(x_j) = \begin{cases} js - \dfrac{j(j-1)}{2} & \text{if } k(x_s) = 1 \\ \dfrac{js - j(j-1)}{2} & \text{if } k(x_j) = 2 \end{cases}$$

$\circ\ \gamma_M^F = \gamma_m^F = s$ or $\gamma_M^F = \gamma_m^F = s/2$

$$\Longrightarrow h(P_n) = \min\{1/s, s = 1, \cdots, \lfloor n/2 \rfloor\} = 1/\lfloor n/2 \rfloor$$

Some examples

✓ The infinite k-regular tree T_k:

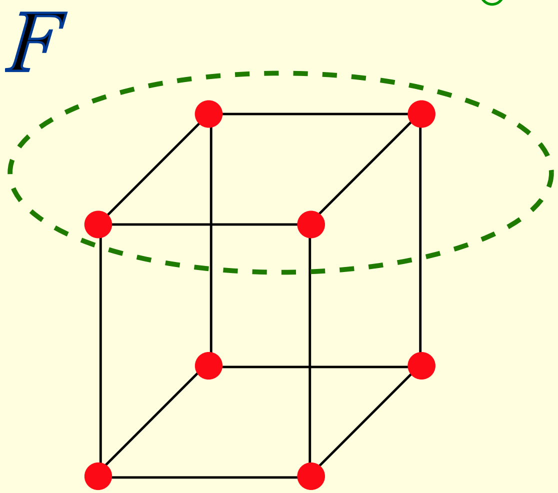


- $\gamma^{B_r}(x) = \dfrac{|B_r|}{2(k-2)+|B_r|}, \quad |x| = r$

- $h(T_k) = k - 2$

Some example

✓ The n-cube Q_n: $|\partial F| \geq \log_2 \left(\frac{2^n}{|F|} \right) |F|$

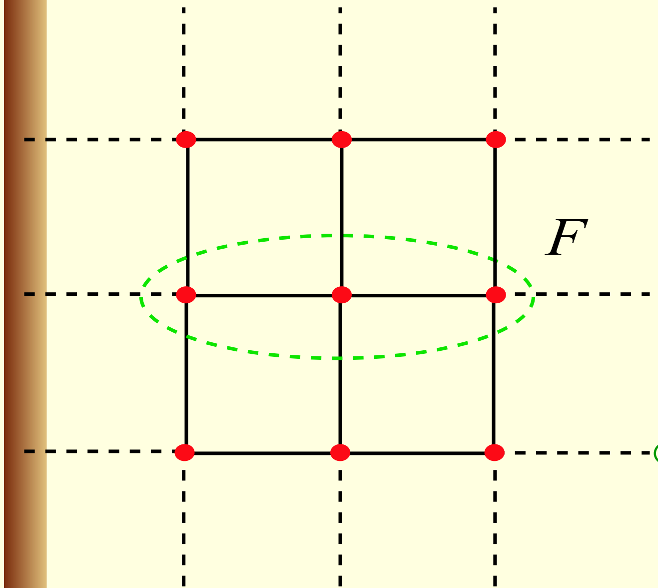


- $\gamma^{B_r}(x) = \frac{|B_r(y)|}{|\partial B_r(y)|} = \frac{\binom{n}{0} + \ldots + \binom{n}{r}}{\binom{n}{r}(n-r)}, \quad |x| = r$

- $F = Q_{n-1}$, $\gamma^F = \chi_F$ and $h(Q_n) = 1$

Some example

✓ The lattice Z^n, $n \geq 2$: $|\partial F| \geq |F| \dfrac{2n}{\sqrt[n]{|F|}}$



○ $\gamma^{B_1}(x) = \dfrac{2n+1}{2n(2n-1)}, \quad |x| = 1$

○ $\gamma^{B_2(y)}(x) = \begin{cases} \dfrac{4n^2+2n+3}{2n(4n^2-6n+3)}, & \text{if } |x-y| = \sqrt{2} \\ \dfrac{4n^2-2n+3}{2n(4n^2-6n+3)}, & \text{if } |x-y| = 2 \end{cases}$

○ $\dfrac{2n(4n^2-6n+3)}{4n^2+2n+3} \leq \dfrac{|\partial B_2(y)|}{|B_2(y)|}$

○ $\dfrac{2n}{\sqrt[n]{2n(n+1)+1}} \leq \dfrac{|\partial B_2(y)|}{|B_2(y)|}$

Eigenvalues BVP

- $F \subset V, \delta(F) = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$:

$$\begin{aligned} \mathcal{L}(u)(x) &= \lambda\, u(x), & \text{if } x \in F \\ \frac{\partial u}{\partial \mathsf{n}_F}(x) &= 0, & \text{if } x \in H_1 \\ u(x) &= 0, & \text{if } x \in H_2 \end{aligned}$$

Eigenvalues BVP

- $F \subset V, \delta(F) = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$:

$$\begin{aligned} \mathcal{L}(u)(x) &= \lambda\, u(x), && \text{if } x \in F \\ \frac{\partial u}{\partial \mathsf{n}_F}(x) &= 0, && \text{if } x \in H_1 \\ u(x) &= 0, && \text{if } x \in H_2 \end{aligned}$$

$$\lambda(F, H_1, H_2) = \min_{\substack{u \in \mathcal{C}(F \cup H_1) \\ u \neq 0}} \left\{ \frac{\displaystyle\int_F u\, \mathcal{L} u\, \nu\, dx}{\displaystyle\int_F u^2 \nu\, dx} : \quad \frac{\partial u}{\partial \mathsf{n}_F} = 0 \text{ on } H_1 \;\; a \int_F u\nu\, dx = 0 \right\}$$

$a = 1$ if $H_2 = \emptyset$ Neumann and Poisson problems

$a = 0$ otherwise

Poisson eigenvalues

- Neumann problem can be reduced to a Poisson problem on a new network

Poisson eigenvalues

$$\mathcal{L}(u)(x) = \lambda\, u(x), \text{ if } x \in V$$

Poisson eigenvalues

$$\mathcal{L}(u)(x) = \lambda\, u(x), \text{ if } x \in V$$

$$\lambda(\Gamma) = \min_{\substack{u \in \mathcal{C}(V) \\ u \neq 0}} \left\{ \frac{\displaystyle\int_V u\, \mathcal{L}u\, \nu\, dx}{\displaystyle\int_V u^2 \nu\, dx} : \int_V u\nu\, dx = 0 \right\}$$

Poisson eigenvalues

$$\mathcal{L}(u)(x) = \lambda\, u(x), \text{ if } x \in V$$

$$\lambda(\Gamma) = \min_{\substack{u \in \mathcal{C}(V) \\ u \neq 0}} \left\{ \frac{\int_V u\, \mathcal{L} u\, \nu\, dx}{\int_V u^2 \nu\, dx} : \int_V u\nu\, dx = 0 \right\}$$

$\Longrightarrow$ If γ_x is the equilibrium measure for the set $V \setminus \{x\}$, then

$$\min_{x \in V} \left\{ \frac{\mathrm{vol}_\nu(V)}{||\gamma_x||_{1,\nu}} \right\} \leq \lambda(\Gamma) \leq \min_{x \in V} \left\{ \frac{\mathrm{vol}_\nu(V) ||\gamma_x||_{1,\nu}}{\mathrm{vol}_\nu(V) ||\gamma_x||^2_{2,\nu} - ||\gamma_x||^2_{1,\nu}} \right\}$$

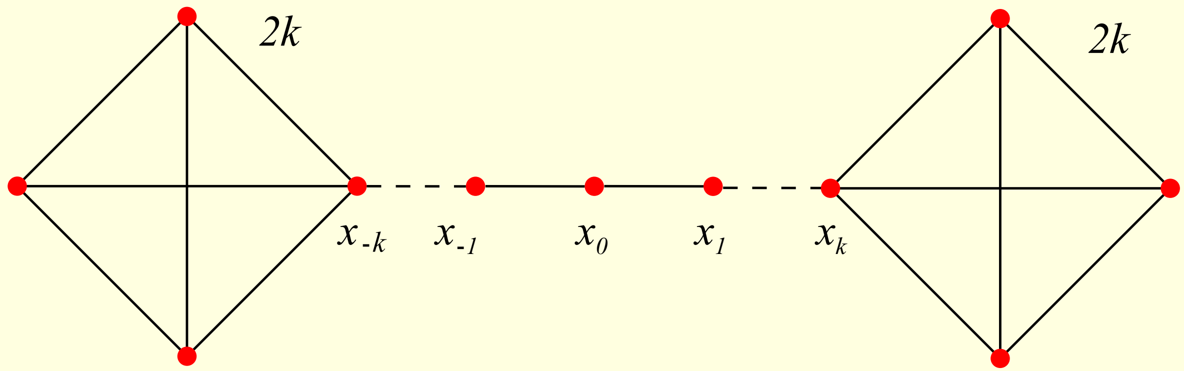
Poisson eigenvalues: sharpness

- The path P_n: $\lambda(\Gamma) = 2 - 2\cos\left(\frac{\pi}{n}\right) \sim \dfrac{\pi^2}{n^2}$

$$\frac{6}{(n-1)(2n-1)} \le \lambda(\Gamma) \le \frac{30}{(n+1)(2n+1)}, \quad n \ge 2$$

Poisson eigenvalues: sharpness

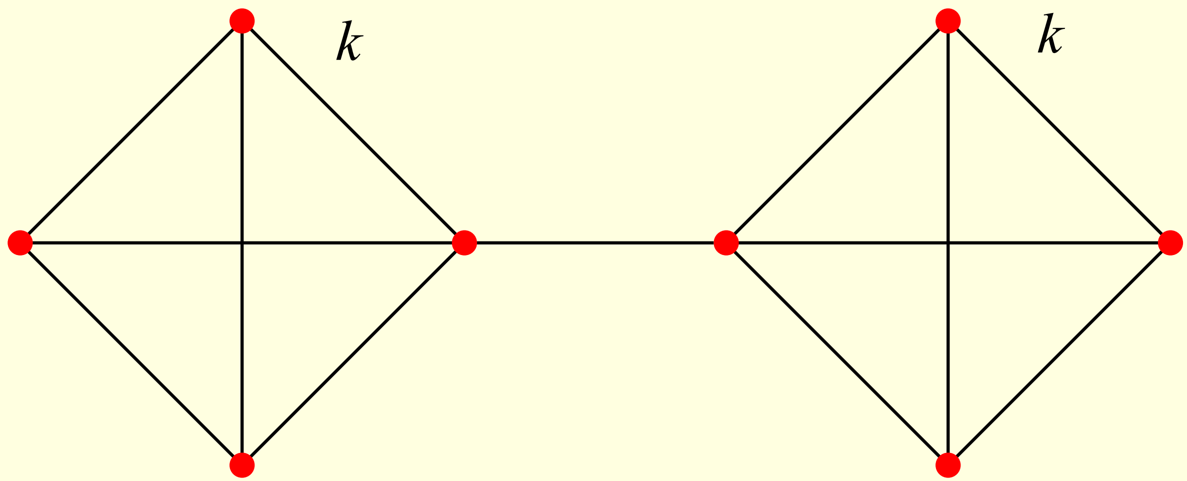
- The bar-bell $B_n, n = 6k - 1$: $\lambda(\Gamma) \in \mathcal{O}(n^{-2})$



$$\frac{81}{7n^2} \sim \lambda(\Gamma) \sim \frac{2835}{178n^2}, \quad n \geq 23$$

Poisson eigenvalues: sharpness

- The dum-bell $D_n, n = 2k$: $\lambda(\Gamma) \in \mathcal{O}(n^{-1})$



$$\frac{4n}{5n^2 - 4n - 8} \leq \lambda(\Gamma) \leq \frac{4n(n^2 + 32n - 64)}{n^4 + 32n - 64}$$

Some examples

- Distance regular graph

$$\frac{|V|}{\sum_{j=0}^{D-1} \frac{(|V|-|B_j|)^2}{|\partial B_j|}} \leq \lambda(\Gamma) \leq \frac{|V| \sum_{j=0}^{D-1} \frac{(|V|-|B_j|)^2}{|\partial B_j|}}{\sum_{j=0}^{D-1} \frac{|B_j|\,(|V|-|B_j|)^3}{|\partial B_j|^2} + 2 \sum_{0\leq i<j\leq D-1} \frac{|B_i|\,(|V|-|B_i|)(|V|-|B_j|)^2}{|\partial B_i||\partial B_j|}}$$

Dirichlet Eigenvalues

$$\begin{aligned} \mathcal{L}(u)(x) &= \lambda\, u(x), && \text{if } x \in F \\ u(x) &= 0, && \text{if } x \in \delta(F) \end{aligned}$$

Dirichlet Eigenvalues

$$\begin{aligned} \mathcal{L}(u)(x) &= \lambda\, u(x), & \text{if } x \in F \\ u(x) &= 0, & \text{if } x \in \delta(F) \end{aligned}$$

$$\lambda_d(F) = \min_{\substack{u \in \mathcal{C}(F) \\ u \neq 0}} \left\{ \frac{\int_F u\, \mathcal{L} u\, \nu\, dx}{\int_F u^2 \nu\, dx} \right\}$$

Dirichlet Eigenvalues

$$\begin{aligned} \mathcal{L}(u)(x) &= \lambda\, u(x), \quad && \text{if } x \in F \\ u(x) &= 0, && \text{if } x \in \delta(F) \end{aligned}$$

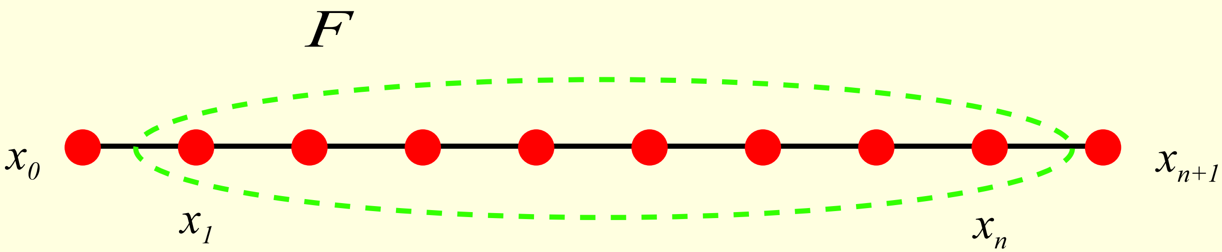
$$\lambda_d(F) = \min_{\substack{u \in \mathcal{C}(F) \\ u \neq 0}} \left\{ \frac{\displaystyle\int_F u\, \mathcal{L} u\, \nu\, dx}{\displaystyle\int_F u^2 \nu\, dx} \right\}$$

$\Longrightarrow$ If γ^F is the equilibrium measure for F

$$\min_{x \in F} \left\{ \frac{1}{\gamma^F(x)} \right\} \le \lambda_d(F) \le \frac{||\gamma^F||_{1,\nu}}{||\gamma^F||_{2,\nu}^2}$$

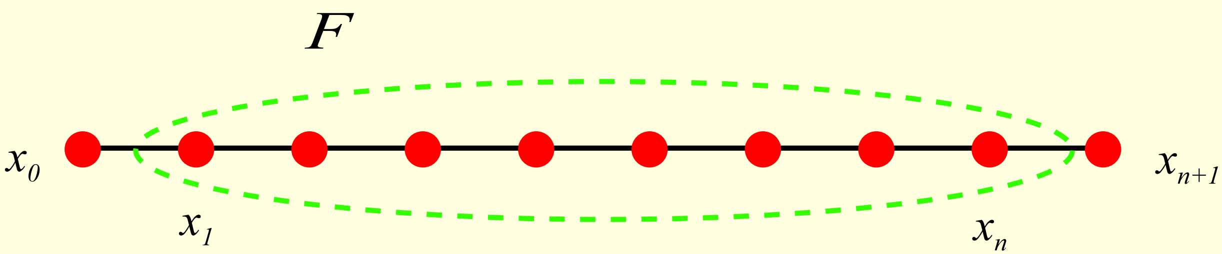
Some examples

- P_{n+2}, $\nu = 2 \cdot \mathbf{1}$ $\quad \gamma^F(x_i) = \dfrac{i(n+1) - i^2}{2}$, $i = 1, \ldots, n$



Some examples

- P_{n+2}, $\nu = 2 \cdot 1$ $\quad \gamma^F(x_i) = \dfrac{i(n+1) - i^2}{2}, \; i = 1, \ldots, n$



$$\lambda_d(F) = 2 - 2\cos\left(\frac{\pi}{n+1}\right) \sim \frac{\pi^2}{(n+1)^2}$$

$$\frac{2}{\left\lceil \frac{(n+1)^2}{4} \right\rceil} \leq \lambda_d(F) \leq \frac{10}{(n+1)^2 + 1}$$

Some examples

- Distance regular graph

$$\frac{1}{\displaystyle\sum_{s=0}^{r}\frac{|B_s|}{|\partial B_s|}} \leq \lambda_d(B_r) \leq \frac{\displaystyle\sum_{i=0}^{r} k_i \sum_{s=i}^{r} \frac{|B_s|}{|\partial B_s|}}{\displaystyle\sum_{i=0}^{r} k_i \left(\sum_{s=i}^{r} \frac{|B_s|}{|\partial B_s|}\right)^2}$$